STRUCTURE OF RELAXATION ZONES BEHIND A SHOCK FRONT IN CHEMICALLY ACTIVE GAS MIXTURES

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We consider substances with an arbitrary number of chemical reactions on the assumption that the frozen and equilibrium velocities of sound are close to each other in magnitude. The velocity of an oncoming stream is considered higher than the velocity of propagation of small oscillations in a mixture with frozen composition. Under this condition, the boundary of the unperturbed flow is constituted by a shock front behind which there are relaxation zones. By the method of joining the external and internal asymptotic expansions, we investigate how their structure varies with the increasing velocity of the stream at infinity.

1. Initial Equations. We use the subscript ∞ to characterize a substance in the unperturbed state. We shall assume that the frozen velocity of sound $a_{f\infty}$ and the equilibrium velocity of sound $a_{e\infty}$ are close to each other in magnitude and that their difference is proportional to a small parameter ε_a^2. We assume that the velocity v_∞ of the oncoming stream does not deviate greatly from any of the so-called intermediate velocities $\alpha_{k\infty}$ of propagation of acoustic waves. Using the numbers $\gamma^{(k)}$ to represent this deviation, we have

$$v_\infty - \alpha_{k\infty} = \varepsilon_a^2 \gamma^{(k)} v_\infty. \tag{1.1}$$

In a mixture in which N reactions can take place simultaneously, the subscript k runs through the values 0, 1, ..., N. The intermediate velocities of sound satisfy the inequalities [1-3]

$$a_{e\infty} = \alpha_{0,\infty} < \alpha_{1,\infty} < \ldots < \alpha_{N-1,\infty} < \alpha_{N,\infty} = a_{f\infty}. \tag{1.2}$$

In the limiting cases k = 0 and k = N, the constants $\gamma^{(0)} = \gamma_e$ and $\gamma^{(N)} = \gamma_f$.

In order to study the relaxation zones situated behind a weak shock front in a multicomponent mixture, we shall make use of a system of equations which describes stationary one-dimensional flow in the transonic velocity range [4]:

$$2\left(\varepsilon m_\infty v' + \varepsilon_a^2 \gamma_f\right) \frac{dv'}{dx'} = \delta_a^2 \mathbf{e}_2' \frac{d\mathbf{q}_2'}{dx'}, \quad \delta_a^2 = \frac{p_\infty}{\rho_\infty v_\infty^2} \varepsilon_a^2, \quad \frac{d\mathbf{q}_2'}{dx'} = -\mathbf{E}\boldsymbol{\omega}_2', \tag{1.3}$$
$$\boldsymbol{\omega}_2' = \mathbf{D}\mathbf{q}_2' + \mathbf{e}_2 v'.$$

The coordinate x', the desired velocity v' of perturbed particle motion, and the components of the vectors $\mathbf{q}_2' = (q_{21}', \ldots, q_{2N}')$ and $\boldsymbol{\omega}_2' = (\omega_{21}', \ldots, \omega_{2N}')$ representing the completeness and affinity of the chemical reactions are taken here in a special dimensionless system of units. The letters ρ, p, and m denote the density, the pressure, and a dimensionless thermodynamic coefficient proportional to the curvature of the Poisson adiabatic curve for a mixture of constant composition. The small parameter ε is a measure of the amplitude of the perturbations.

In the initial Euler equations any positive-definite and symmetric matrices may appear as the kinetic matrix and the stability matrix of the system. The linear transformations of the completeness and affinity vectors of the chemical reactions enable us to reduce these matrices to a unit matrix **E** and a diagonal matrix **D**, respectively. In the system of equations (1.3) this transformation is assumed to have been carried out. The components of the

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constant vector $e_2' = (e_{21}', \ldots, e_{2N}')$, which are calculated by means of the adiabatic derivatives of the specific internal energy of the system per unit volume and one of the components of the completeness vector, are also assumed to have been subjected to the above-mentioned linear transformations.

Suppose that the coordinate of the shock front is $x' = 0$. In front of it the oncoming stream remains uniform and is in a state of complete thermodynamic equilibrium, i.e., $v' = 0$, $\mathbf{q}_2' = 0$ for $x' < 0$. In the process of shock compression the composition of the mixture cannot change. Denoting the difference between the values of any parameter of the gas before and after the shock wave by square brackets, we have $[q_{2i}'] = 0$, $i = 1, \ldots, N$.

The change in the velocity upon passing through the shock front can be calculated as follows. The result of integrating the first of the equations in (1.3) is

$$\frac{1}{\varepsilon m_\infty}(\varepsilon m_\infty v' + \varepsilon_a^2 \gamma_f)^2 = \delta_a^2 \mathbf{e}_2' \mathbf{q}_2' + C,$$

where C is an arbitrary constant. Applying the resulting equation to points situated on opposite sides of the surface of discontinuity, we find $[(\varepsilon m_\infty v' + \varepsilon_a^2 \gamma_f)^2] = 0$.

The initial data for integrating the system of equations (1.3) take the form

$$v' = -2\frac{\varepsilon_a^2 \gamma_f}{\varepsilon m_\infty}, \quad q_{2i}' = 0, \quad i = 1, \ldots, N \text{ for } x' = 0. \tag{1.4}$$

By Cemplen's theorem, the velocity of the gas is reduced by the shock compression, and therefore behind the surface of discontinuity $v' < 0$. The last requirement can be satisfied if $\gamma_f > 0$. On the basis of the definition (1.1), we arrive at the well-known conclusion that shock waves are formed when the velocity of the stream exceeds not only the equilibrium velocity but also the frozen velocity of sound.

After compression, inside the relaxation zones the gas mixture reaches a new equilibrium state, and therefore

$$v' \to v_0', \quad \frac{dv'}{dx'} \to 0, \quad \frac{d\mathbf{q}_2'}{dx'} \to 0 \text{ as } x' \to \infty. \tag{1.5}$$

The value $C = \varepsilon_a^4 \gamma_f^2 / (\varepsilon m_\infty)$ of the constant of integration is found to be the same as for the continuous flows which take place when the velocity of the oncoming stream lies in the range between the equilibrium and frozen velocities of sound. Using the boundary conditions (1.5), we obtain the relation [5]

$$v_0' = -2\frac{\varepsilon_a^2 \gamma_e}{\varepsilon m_\infty}$$

between the velocity of the particles after passing through the relaxation zones and the coefficient γ_e or γ_f.

2. Intermediate Velocities of Sound. For intermediate velocities of propagation of small oscillations the following formula holds [1-3]:

$$\alpha_{k\infty} = a_{f\infty} + \frac{1}{2}\delta_a^2 v_\infty \sum_{m=k+1}^{N} (-1)^{m-k} \frac{\sigma_{N-m}}{\sigma_{N-k}} \mathbf{e}_2' \mathbf{D}^{m-k-1} \mathbf{e}_2'. \tag{2.1}$$

Here the symbol σ_l denotes the sum of all possible products made up of the positive eigenvalues $\lambda_1, \ldots, \lambda_N$, equal respectively to the diagonal elements $d_{11}, \ldots, d_{NN}$ of the relaxation matrix $\mathbf{R} = \mathbf{ED} = \mathbf{D}$, taken l at a time in each product.

We introduce the auxiliary diagonal matrix

$$\mathbf{D}^{(k)} = \sum_{m=k+1}^{N} (-1)^{m-k} \frac{\sigma_{N-m}}{\sigma_{N-k}} \mathbf{D}^{m-k-1}$$

with elements

$$d_{ii}^{(k)} = \sum_{m=k+1}^{N} (-1)^{m-k} \frac{\sigma_{N-m}}{\sigma_{N-k}} \lambda_i^{m-k-1} = -\frac{\sigma_{N-k-1}^{(i)}}{\sigma_{N-k}},$$

where the upper subscript (i) in the sum $\sigma_l^{(i)}$ indicates that λ_i is excluded from the complete set of eigenvalues $\lambda_1,\ldots, \lambda_N$ appearing in it.

We assume, as often happens with real chemically active systems, that all the relaxation processes can be subdivided into two groups. Suppose that one of these is formed by M slow reactions and that the remaining N − M reactions are fast. Then

$$\lambda_1, \ldots, \lambda_{N-M} >> \lambda_{N-M+1}, \ldots, \lambda_N. \tag{2.2}$$

The approximate expression for the M-th intermediate velocity of sound has the form [6]

$$\alpha_{M\infty} = a_{f\infty} - \frac{1}{2} \delta_a^2 v_\infty \sum_{l=1}^{N-M} \frac{e_{2l}'^2}{\lambda_l} \tag{2.3}$$

if each of the ratios $e_{2i}'^2/\lambda_i$, i = 1,..., N is comparable to unity in order of magnitude.

Let us find the asymptotic expressions determining the other velocities of propagation of small perturbations. We consider the case k > M. The principal terms of the elements of the auxiliary matrix $\mathbf{D}^{(k)}$ will be

$$d_{ii}^{(k)} = -\frac{\sigma_{N-k-1}^{(i,N-M+1,\ldots,N)}}{\sigma_{N-k}^{(N-M+1,\ldots,N)}} \quad \text{for} \quad 1 \leqslant i \leqslant N-M,$$

$$d_{ii}^{(k)} = -\frac{\sigma_{N-k-1}^{(N-M+1,\ldots,N)}}{\sigma_{N-k}^{(N-M+1,\ldots,N)}} \quad \text{for} \quad N-M+1 \leqslant i \leqslant N.$$

Substituting these equations into Eq. (2.1), we obtain

$$\alpha_{k\infty} = a_{f\infty} - \frac{1}{2} \delta_a^2 v_\infty \sum_{l=1}^{N-M} e_{2l}'^2 \frac{\sigma_{N-k-1}^{(l,N-M+1,\ldots,N)}}{\sigma_{N-k}^{(N-M+1,\ldots,N)}}. \tag{2.4}$$

It can be seen that the relation so obtained coincides with the initial relation if the eigenvalues λ_i(i = N − M + 1,..., N) are formally set equal to zero. Thus, the k-th intermediate velocity of sound for k > M can be calculated by the exact formula (2.1), regarding all the slow reactions as frozen.

On the other hand, when k < M, the simplified expressions for the elements of the auxiliary matrix $\mathbf{D}^{(k)}$ can be written in the form

$$d_{ii}^{(k)} = -\frac{1}{\lambda_i} \quad \text{for} \quad 1 \leqslant i \leqslant N-M,$$

$$d_{ii}^{(k)} = -\frac{\sigma_{M-k-1}^{(1,\ldots,N-M,i)}}{\sigma_{M-k}^{(1,\ldots,N-M)}} \quad \text{for} \quad N-M+1 \leqslant i \leqslant N.$$

In the case under consideration, it follows from Eq. (2.1) that

$$\alpha_{k\infty} = a_{f\infty} - \frac{1}{2} \delta_a^2 v_\infty \sum_{l=1}^{N-M} \frac{e_{2l}'^2}{\lambda_l} - \frac{1}{2} \delta_a^2 v_\infty \sum_{l=N-M+1}^{N} e_{2l}'^2 \frac{\sigma_{M-k-1}^{(1,\ldots,N-M,l)}}{\sigma_{M-k}^{(1,\ldots,N-M)}}.$$

From the definition (2.3),

$$\alpha_{k\infty} = \alpha_{M\infty} - \frac{1}{2} \delta_a^2 v_\infty \sum_{l=N-M+1}^{N} e_{2l}'^2 \frac{\sigma_{M-k-1}^{(1,\ldots,N-M,l)}}{\sigma_{M-k}^{(1,\ldots,N-M)}}. \tag{2.5}$$

The last equation is completely analogous to the initial equation, but in this equation the role of the frozen velocity is played by the M-th intermediate velocity of propagation of acoustic waves. As a result, when we set up the sums $\sigma_{N-k}^{(1,\ldots,N-M)}$ and $\sigma_{M-k-1}^{(1,\ldots,N-M,l)}$, we do not make use of the eigenvalues $\lambda_1,\ldots,\lambda_{N-M}$, which belong to the fast relaxation processes, although at first glance the result obtained in this way seems paradoxical.

In what follows, we shall make use of the following inequality, which is stronger than (2.2):

$$\lambda_1 >> \lambda_2 >> \ldots >> \lambda_{N-1} >> \lambda_N. \tag{2.6}$$

The assumption $e_{2i}'^2/\lambda_i \sim 1$, $i = 1,\ldots,N$, is assumed to remain valid. The inequalities (2.6) characterize systems in which the rates of all relaxation processes differ in order of magnitude. Systems of this kind are also often encountered in practice.

It is evident that now the M-th velocity will not play any special role among the other intermediate velocities of sound. In fact, for the sums appearing on the right-hand side of formula (2.4), we have the asymptotic estimates

$$\begin{aligned}
&\sigma_{N-k-1}^{(l,N-M+1,\ldots,N)} \sim \lambda_1 \ldots \lambda_{l-1}\lambda_{l+1}\lambda_{N-k} \quad \text{for} \quad 1 \leqslant l \leqslant N-k-1,\\
&\sigma_{N-k-1}^{(l,N-M+1,\ldots,N)} \sim \lambda_1 \ldots \lambda_{N-k-1} \quad \text{for} \quad N-k \leqslant l \leqslant N-M,\\
&\sigma_{N-k}^{(N-M+1,\ldots,N)} \sim \lambda_1 \ldots \lambda_{N-k}.
\end{aligned} \tag{2.7}$$

For $k > M$ the difference $N - k < N - M$, and from this we obtain the relation (2.3), in which the subscript M has been replaced by k.

Estimates analogous to (2.7) may also be written for the sums $\sigma_{M-k-1}^{(1,\ldots,N-M,l)}$, and $\sigma_{M-k}^{(1,\ldots,N-M)}$, appearing in formula (2.5). Since $N - k > N - M$, if $k < M$, the result is that we again obtain the relation (2.3):

$$\alpha_{k\infty} = \alpha_{M\infty} - \frac{1}{2}\delta_a^2 v_\infty \sum_{l=N-M+1}^{N-k} \frac{e_{2l}'^2}{\lambda_l},$$

where on the right-hand side the summation is carried out for numbers l in the interval $N - M + 1 \leqslant l \leqslant N - k$.

For the sake of brevity, in our notation for both constant and variable quantities, we shall hereafter omit the primes and the index 2, which indicates the result of a linear transformation of the completeness and affinity vectors of the chemical reactions.

3. Monotonicity Properties. It has long been known [7, 8] that shock compression of a chemically active mixture results in the appearance of relaxation zones in the stream. If the rates of the reactions differ substantially in magnitude, these zones will have different widths and will be arranged one behind the other. The width of each zone is determined by one or more relaxation processes. The calculations of [9] confirm the conclusion that there is a "band" structure in the stream which has passed through the shock front and approaches a new equilibrium state. The solution obtained in [10] for the problem of the motion of a plane piston in a gas mixture enables us to judge how the relaxation zones with different widths are arranged with respect to time.

The assumption made at the outset that the frozen and equilibrium velocities of sound are close to each other in magnitude considerably simplifies the analysis of the nonlinear system of equations (1.3). The use of the method of joining the external and internal asymptotic expansions made it possible to establish [5] all the laws associated with the "band" structure of shock waves with complete dispersion (i.e., those not including any surfaces of discontinuity). On the basis of the results obtained in [5], we can construct the flow in the relaxation zones behind the surface of discontinuity, on passing through which the gas is subjected to a sudden compression.

First of all, let us prove that the thermodynamic parameters $q_i^x = \lambda_i q_i/e_i$ are monotone decreasing along the coordinate x. Regarding v as a known function and taking account of the equation x = 0, which determines the position of the shock front, we have

$$q_i^x = -\lambda_i \int_0^x v(\xi)\, e^{\lambda_i(\xi - x)} d\xi, \quad i = 1, \ldots, N. \tag{3.1}$$

We substitute the initial values (1.4) of the desired functions into the system of equations (1.3) and calculate their derivatives on the surface of discontinuity. Since the parameters of the mixture can undergo jump discontinuities only when $\gamma_f > 0$, it follows that

$$\frac{dq_i^x}{dx} = -2\,\frac{\varepsilon_a^2 \lambda_i \gamma_f}{\varepsilon m_\infty} > 0, \quad i = 1, \ldots, N, \quad \frac{dv}{dx} = -\frac{\delta_a^2}{\varepsilon m_\infty} \sum_{i=1}^{N} e_i^2 < 0.$$

The second of these inequalities leads to the assertion that there exists some range $0 \leq x \leq x_0$, in which the derivative dv/dx < 0. The monotonic variation of v will be established if we find that $dv(x_0)/dx$ is a negative quantity.

Assume the contrary and set $dv(x_0)/dx = 0$. Then, in accordance with the first equation of the system (1.3), we have at the point x_0

$$\sum_{i=1}^{N} \frac{e_i^2}{\lambda_i} \frac{dq_i^x(x_0)}{dx} = 0. \tag{3.2}$$

Combining the remaining equations of this system with the expressions (3.1), we obtain

$$\frac{dq_i^x(x_0)}{dx} = -\lambda_i \left\{ v(x_0)\, e^{\lambda_i x_0} + \lambda_i \int_0^{x_0} [v(x_0) - v(\xi)]\, e^{\lambda_i(\xi - x_0)}\, d\xi \right\}, \quad i = 1, \ldots, N.$$

The function v(x) attains its minimum value $v(x_0) < 0$ at the boundary of the interval $0 \leqslant x \leqslant x_0$. From this it follows at once that both of the terms in braces on the right-hand side of the last relation are negative, and the derivatives

$$\frac{dq_i^x(x_0)}{dx} > 0, \quad i = 1, \ldots, N. \tag{3.3}$$

Summing inequalities (3.3), after first multiplying them by e_i^2/λ_i, we arrive at a contradiction of formula (3.2). This contradiction proves that the velocity of the mixture decreases monotonically as the coordinate x increases. Finally, in order to conclude that the components of the vector $q^x = (q_1^x, \ldots, q_N^x)$ are monotone increasing, it is sufficient to consider once more the equations for $dq_i^x(x_0)/dx$.

4. Transition through the Frozen Velocity of Sound. We begin with the limiting case $\gamma_f = 0$. According to the Cauchy data (1.4), we have

$$v = 0; \quad q_i = 0, \; i = 1, \ldots, N \text{ for } x = 0. \tag{4.1}$$

In other words, the velocity and all the thermodynamic functions remain continuous. The derivatives $dq_i/dx = 0$, i.e., are also continuous, but, as will be seen later, $[dv/dx] \neq 0$. The jump in the derivative of the velocity means that the point x = 0 corresponds to the characteristic of the partial differential equations governing the flow of relaxation mixtures.

In order to construct the field of perturbations in the relaxation zone adjacent to the characteristic, we introduce a new scale of length by means of the formula

$$x = x_1/\lambda_1. \tag{4.2}$$

The first equation of the initial system (1.3) becomes

$$2\varepsilon m_\infty v\frac{dv}{dx_1} = -\delta_a^2\sum_{i=1}^{N}\frac{1}{\lambda_1}e_i(\lambda_i q_i + e_i v). \tag{4.3}$$

The remaining equations, taking account of the requirements $\lambda_k/\lambda_1 \to 0$, $k = 2,\ldots,N$, yield

$$\frac{dq_1}{dx_1} = -\left(q_1 + \frac{e_1}{\lambda_1}v\right),\quad \frac{dq_k}{dx_1} = 0,\quad k = 2,\ldots,N. \tag{4.4}$$

It is clear that we can simplify Eq. (4.3), replacing it with

$$2\varepsilon m_\infty v\frac{dv}{dx} = \delta_a^2 e_1\frac{dq_1}{dx_1}. \tag{4.5}$$

Equations (4.4), (4.5) form a closed system signifying that in the region under consideration all the reactions except the first one are in the frozen state.

The resulting system is equivalent to a single second-order equation

$$\varepsilon m_\infty\frac{d}{dx_1}\left(v\frac{dv}{dx_1}\right) + \left(\varepsilon m_\infty v + \varepsilon_a^2\gamma^{(N-1)}\right)\frac{dv}{dx_1} = 0, \tag{4.6}$$

since the numbers [6, 10]

$$\gamma^{(k)} = \gamma_f + \frac{1}{2}\frac{p_\infty}{\rho_\infty v_\infty^2}\sum_{l=1}^{N-k}\frac{e_l^2}{\lambda_l}. \tag{4.7}$$

The numbering of these numbers corresponds, of course, to the numbering given by the inequalities (1.2). The integral of Eq. (4.6) satisfying the initial data (4.1) has the form

$$v = -2\frac{\varepsilon_a^2\gamma^{(N-1)}}{\varepsilon m_\infty}\left[1 - \exp\left(-\frac{1}{2}x_1\right)\right]. \tag{4.8}$$

At the point x = 0 the value of the derivative

$$\frac{dv}{dx} = -\frac{1}{2}\frac{\delta_a^2 e_1^2}{\varepsilon m_\infty}$$

is different from the value found when the gas is subjected to shock compression. However, for $\gamma_f = 0$ the first of the equations (1.3) is satisfied for any value of dv/dx.

Now let $\gamma_f > 0$, i.e., let the shock wave serve as the boundary of the perturbed stream. The scale of the first relaxation zone is determined, as before, by Eq. (4.2). Equations (4.4) remain valid, but instead of Eq. (4.5) we must write

$$2\left(\varepsilon m_\infty v + \varepsilon_a^2\gamma_f\right)\frac{dv}{dx_1} = \delta_a^2 e_1\frac{dq_1}{dx_1}. \tag{4.9}$$

As in the limiting case $\gamma_f = 0$, the system of equations (4.4), (4.9) corresponds to a situation in which only the first reaction goes on, while the remaining N − 1 relaxation processes are practically frozen. The second-order equation equivalent to this system has the form

$$\frac{d}{dx_1}\left[\left(\varepsilon m_\infty v + \varepsilon_a^2\gamma_f\right)\frac{dv}{dx_1}\right] + \left(\varepsilon m_\infty v + \varepsilon_a^2\gamma^{(N-1)}\right)\frac{dv}{dx_1} = 0.$$

Its integral, satisfying the conditions

$$v = -2\frac{\varepsilon_a^2\gamma_f}{\varepsilon m_\infty}, \quad \frac{dv}{dx_1} = -\frac{\delta_a^2}{\varepsilon m_\infty}\frac{e_1^2}{\lambda_1} \quad \text{for} \quad x_1 = 0,$$

will be

$$-x_1 = \frac{\gamma_f}{\gamma^{(N-1)}}\ln\left|\frac{\varepsilon m_\infty v}{2\varepsilon_a^2\gamma_f}\right| + \left[2 - \frac{\gamma_f}{\gamma^{(N-1)}}\right]\ln\left|\frac{\varepsilon m_\infty v + 2\varepsilon_a^2\gamma^{(N-1)}}{2\varepsilon_a^2\left[\gamma^{(N-1)} - \gamma_f\right]}\right|. \tag{4.10}$$

As $\gamma_f \to 0$, this formula becomes (4.8).

The integral (4.10) can be used for any value of the particle velocity attained when the stream is slowed down by the shock wave. In [3, 4] we discussed in detail the concept of the M-fold frozen and (N − M) -fold equilibrium velocity of sound. For the conditions given by the inequalities (2.6), this velocity of sound in the unperturbed state is simply the M-th intermediate velocity of propagation of small perturbations along an oncoming uniform stream. Taking account of formula (4.7), in the initial dimensional variables the M-fold frozen and (N − M)-fold equilibrium velocity of sound is given by the expression

$$a_{fe}^{(M)} = v_\infty\left[1 - 2\frac{\varepsilon_a^2\gamma_f}{m_\infty} + \varepsilon_a^2\left(\gamma_f - \frac{1}{2}\frac{p_\infty}{\rho_\infty v_\infty^2}\sum_{l=1}^{N-M}\frac{e_l^2}{\lambda_l}\right)\right]$$

for points situated directly behind the shock-front. In the same variables, the velocity of the particles is

$$v = v_\infty\left(1 - 2\varepsilon_a^2\frac{\gamma_f}{m_\infty}\right).$$

From this it can be seen at once that

$$v < a_{fe}^{(M)} \quad \text{for} \quad \gamma_f > \frac{1}{2}\frac{p_\infty}{\rho_\infty v_\infty^2}\sum_{l=1}^{N-M}\frac{e_l^2}{\lambda_l},$$

$$v > a_{fe}^{(M)} \quad \text{for} \quad \gamma_f < \frac{1}{2}\frac{p_\infty}{\rho_\infty v_\infty^2}\sum_{l=1}^{N-M}\frac{e_l^2}{\lambda_l}.$$

The equation $v = a_{fe}^{(M)}$ for a gas passing through the shock wave can be given the form

$$v_\infty = 2a_{f\infty} - a_{M\infty}. \tag{4.11}$$

The last relation, in particular, shows that as $\gamma_f \to 0$, the shock front degenerates into the characteristic, and the velocity of the particles when they intersect it remains unchanged.

If the quantity γ_f is only slightly greater than zero, then the velocity of sound behind the shock wave is less than the frozen velocity but greater than any of the M-fold frozen and (N − M)-fold equilibrium velocities of sound. As γ_f increases, the particle velocity passes successively through the local velocities of sound $a_{fe}^{(M)}$ with indices M = N − 1, N − 2,...,1. Equation (4.11) specifies each such transition. This equation, in turn, corresponds to the value $\gamma^{(M)} = 2\gamma_f$ in formula (4.7). The velocity of the stream reaches the equilibrium velocity of propagation of small oscillations for the condition

$$\gamma_f = \frac{1}{2}\frac{p_\infty}{\rho_\infty v_\infty^2}\sum_{i=1}^{N}\frac{e_i^2}{\lambda_i},$$

which leads to the value $\gamma_e = 2\gamma_f$. As the velocity of the stream increases further, the velocity of the mixture subjected to shock compression becomes less than the equilibrium velocity of acoustic waves. The qualitative behavior of the solution (4.10) remains the same for any value of γ_f.

5. Sequence of Relaxation Zones. In order to construct the field of flow in the following perturbed region, we introduce the variable

$$x = x_2/\lambda_2. \tag{5.1}$$

The first equation of the initial system (1.3) can be written in the form

$$2\left(\varepsilon m_\infty v + \varepsilon_a^2 \gamma_f\right)\frac{dv}{dx_2} = -\delta_a^2 \sum_{i=1}^{N} \frac{e_i}{\lambda_2}\left(\lambda_i q_i + e_i v\right), \tag{5.2}$$

and from the remaining equations, taking account of the requirements $\lambda_k/\lambda_2 \to 0$, $k = 3,\ldots, N$, we derive

$$\frac{dq_1}{dx_2} = -\frac{\lambda_1}{\lambda_2}\left(q_1 + \frac{e_1}{\lambda_1}v\right), \quad \frac{dq_2}{dx_2} = -\left(q_2 + \frac{e_2}{\lambda_2}v\right), \tag{5.3}$$
$$\frac{dq_k}{dx_2} = 0, \ k = 3, \ldots, N.$$

We eliminate the variable q_1 from the system of equations (5.2), (5.3). Making use of formula (4.7), we have

$$\frac{\lambda_2}{\lambda_1}\frac{d}{dx_2}\left[\left(\varepsilon m_\infty v + \varepsilon_a^2\gamma_f\right)\frac{dv}{dx_2}\right] + \frac{1}{2}\delta_a^2\left(\frac{\lambda_2}{\lambda_1} - 1\right)e_2\frac{dq_2}{dx_2} + \left[\left(\varepsilon m_\infty v + \varepsilon_a^2\gamma^{(N-1)}\right) + \frac{1}{2}\delta_a^2\frac{e_2^2}{\lambda_2}\frac{\lambda_2}{\lambda_1}\right]\frac{dv}{dx_2} = 0.$$

Now we pass to the limit as $\lambda_2/\lambda_1 \to 0$. Since $e_2^2/\lambda_2 \sim 1$, we finally obtain a relatively simple first-order equation. Adjoining to it (5.3), in which we have also passed to the limit as $\lambda_2/\lambda_1 \to 0$, we obtain the closed system

$$2\left(\varepsilon m_\infty v + \varepsilon_a^2\gamma^{(N-1)}\right)\frac{dv}{dx_2} = \delta_a^2 e_2 \frac{dq_2}{dx_2}, \tag{5.4}$$
$$\frac{dq_2}{dx_2} = -\left(q_2 + \frac{e_2}{\lambda_2}v\right), \quad \frac{dq_k}{dx_2} = 0, \quad k = 3, \ldots, N.$$

After constructing its solution, we calculate the thermodynamic variable, q_1 from the relation

$$q_1 = -\frac{e_1}{\lambda_1}v. \tag{5.5}$$

The situation encountered here admits of a simple interpretation: The first reaction, which is the fastest, takes place as an equilibrium reaction, the nature of the second relaxation process is determined by the state of the system, and the remaining N − 2 reactions are frozen.

Now let us determine what initial conditions must be imposed when we integrate system (5.4). To do this, we write the asymptotic expression for the integral (4.10) as $x_1 \to \infty$:

$$v = -2\frac{\varepsilon_a^2\gamma^{(N-1)}}{\varepsilon m_\infty} + b_1 \exp\left[-\frac{\gamma^{(N-1)}}{2\gamma^{(N-1)} - \gamma_f}x_1\right], \tag{5.6}$$

$$b_1 = 2\frac{\varepsilon_a^2\left[\gamma^{(N-1)} - \gamma_f\right]}{\varepsilon m_\infty}\left[\frac{\gamma^{(N-1)}}{\gamma_f}\right]^{-\frac{\gamma_f}{2\gamma^{(N-1)} - \gamma_f}}.$$

As $\gamma_f \to 0$, the asymptotic behavior of (5.5) coincides with the exact solution (4.8) of Eq. (4.6). From the second equation of system (5.4) we derive an asymptotic representation for the thermodynamic variable

$$q_1 = \frac{e_1}{\lambda_1}\left\{2\frac{\varepsilon_a^2\gamma^{(N-1)}}{\varepsilon m_\infty} + \frac{2\gamma^{(N-1)}-\gamma_f}{\gamma^{(N-1)}} b_1 \exp\left[-\frac{\gamma^{(N-1)}}{2\gamma^{(N-1)}-\gamma_f}x_1\right]\right\} + d_1\exp(-x_1)$$

with a new constant d_1. In other words, at the end of the first relaxation zone the thermodynamic variable under consideration approaches its equilibrium value

$$q_1 = -\frac{e_1}{\lambda_1}v_{s1}, \quad v_{s1} = -2\frac{\varepsilon_a^2\gamma^{(N-1)}}{\varepsilon m_\infty}. \tag{5.7}$$

If we calculate the parameters of the particles behind the shock front in the mixture, where the first reaction has already reached equilibrium, the role of the frozen velocity of sound $a_{f\infty}$ will obviously be played by the intermediate velocity $\alpha_{N-1,\infty}$. According to the first formula in (1.4), the quantity v_{s_1} is the perturbed velocity of the stream for such sudden compression.

Now we shall make use of the principle of joining the external and internal asymptotic expansions [11]. Since the scales of (4.2), (5.1) are related by the formula

$$x_2 = \frac{\lambda_2}{\lambda_1}x_1,$$

it follows from the limit conditions (5.7) that

$$v = -2\frac{\varepsilon_a^2\gamma^{(N-1)}}{\varepsilon m_\infty}, \quad q_1 = 2\frac{\varepsilon_a^2\gamma^{(N-1)}}{\varepsilon m_\infty}\frac{e_1}{\lambda_1}, \quad q_k = 0, \\ k = 2, \ldots, N \quad \text{for} \quad x_2 = 0. \tag{5.8}$$

It should be noted that the relation (5.5) coincides in form with the first of the limit conditions (5.7) and the following initial value of q_1 (for $x_2 = 0$). This coincidence is simply explained: At the end of the first relaxation zone the fastest reaction, numbered 1, tends to equilibrium and does not deviate from the equilibrium state in the second relaxation zone.

The initial values of the derivatives corresponding to (5.8) have the form

$$\frac{dv}{dx_2} = -\frac{\delta_a^2}{\varepsilon m_\infty}\frac{e_2^2}{\lambda_2}, \quad \frac{dq_2}{dx_2} = 2\frac{\varepsilon_a^2\gamma^{(N-1)}}{\varepsilon m_\infty}\frac{e_2}{\lambda_2}.$$

The above-described behavior of the second reaction of the system (5.4) can be obtained from Eqs. (4.4), (4.9) by formally replacing the constant γ_f with $\gamma^{(N-1)}$. An analogous situation occurs with the Cauchy data, which are found when we integrate both systems. From this we conclude that the field of perturbations in the second relaxation zone is governed by the integral (4.10) if the constants γ_f and $\gamma^{(N-1)}$ are replaced by $\gamma^{(N-1)}$ and $\gamma^{(N-2)}$, respectively. The formulas (5.6), in which the same substitution has been made, will yield the asymptotic behavior of the solution at the exit from the second relaxation zone, i.e., as $x_2 \to \infty$.

The above process of joining the solutions for different regions can be extended further without any changes. The scale of the j-th zone is introduced by means of

$$x = x_j/\lambda_j.$$

The approximate system of equations can be represented in the form [5]

$$2\left(\varepsilon m_\infty v + \varepsilon_a^2\gamma^{(N-j+1)}\right)\frac{dv}{dx_j} = \delta_a^2 e_j \frac{dq_j}{dx_j}, \tag{5.9}$$

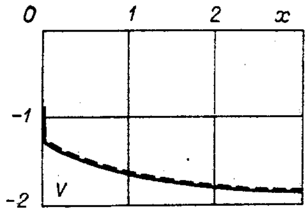

Fig. 1

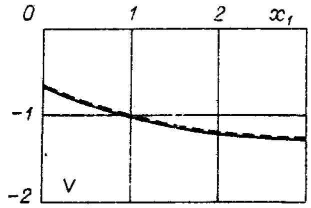

Fig. 2

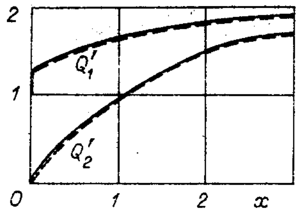

Fig. 3

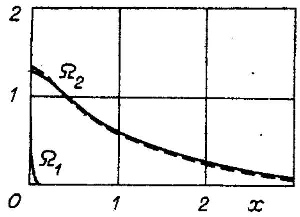

Fig. 4

$$\frac{dq_j}{dx_j} = -\left(q_j + \frac{e_j}{\lambda_j} v\right), \quad \frac{dq_k}{dx_j} = 0, \quad k = j+1, \ldots, N. \tag{5.9}$$

After integrating it, we find that the first $j - 1$ thermodynamic variables $q_1, \ldots, q_{j-1}$ can be reconstructed according to the relations

$$q_i = -\frac{e_i}{\lambda_i} v, \quad i = 1, \ldots, j-1. \tag{5.10}$$

The system of equations (5.9), with the relations (5.10) adjoined to it, can be treated as describing a gas in which a single relaxation process, numbered j, is taking place. The first $j - 1$ faster reactions take place as equilibrium reactions, while the last $N - j$ slower reactions are still frozen. The compression of the mixture ends after the slowest of the reactions goes into an equilibrium regime in the N-th relaxation zone. In each of the N relaxation zones the relaxations can be considered independently, provided that the eigenvalues $\lambda_1, \ldots, \lambda_N$ satisfy inequalities (2.6).

The initial data for the system of equations (5.9) and the relations (5.10) have the form

$$v = -2\frac{\varepsilon_a^2 \gamma^{(N-j+1)}}{\varepsilon m_\infty}, \quad q_i = 2\frac{\varepsilon_a^2 \gamma^{(N-j+1)}}{\varepsilon m_\infty}\frac{e_i}{\lambda_i}, \quad i = 1, \ldots, j-1,$$
$$q_k = 0, \ k = j, \ldots, N \ \text{ for } x_j = 0.$$

The solution of the problem formulated here is obviously given by the expression (4.10), where the role of the constants γ_f and $\gamma^{(N-1)}$ is played by $\gamma^{(N-j+1)}$ and $\gamma^{(N-j)}$. As $x_j \to \infty$ the asymptotic behavior of the field of perturbations is established by the formulas

$$q_j = -\frac{e_j}{\lambda_j} v_{sj}, \quad v_{sj} = -2\frac{\varepsilon_a^2 \gamma^{(N-j)}}{\varepsilon m_\infty},$$

which coincide with (5.7) when $j = 1$. If we calculate the parameters of the particles behind the shock front in a mixture in which the first j reactions have reached equilibrium, the role of the frozen velocity of sound $a_{f\infty}$ will be played by the intermediate velocity $a_{N-j,\infty}$. The quantity v_{sj} is equal to the perturbed velocity of the flow for such sudden compression. Since the width of all the preceding zones approaches zero in the scale of any subsequent zone, continuous compression of the gas in any relaxation zone is seen to be equivalent to compression of the gas at the shock front. When we study the final stage of the process in N-th zone, the first $N - 1$ relaxation zones may be regarded as a sequence of $N - 1$ discontinuities, where in our calculations the frozen velocity of sound $a_{f\infty}$ is replaced in each case by the intermediate velocity $\alpha_{N-j\infty}$, $j = 1 \ldots, N - 1$.

The compression of the mixture in the j-th relaxation zone is accompanied by a change of velocity by an amount

$$\Delta v = -2\frac{\varepsilon_a^2}{\varepsilon m_\infty}\left(\gamma^{(N-j)} - \gamma^{(N-j+1)}\right) = -\frac{\varepsilon_a^2 p_\infty}{\rho_\infty v_\infty^2 \varepsilon m_\infty}\frac{e_j^2}{\lambda_j} < 0,$$

which agrees with the general property, proved in Sec. 3, that it is monotone decreasing as we move away from the shock front.

In conclusion, we consider the difference between continuous flows (completely dispersed shock waves) and motions which include forward-shock discontinuities (shock waves with partial dispersion). The number of relaxation zones in the first type of motion depends on the value of the velocity of the oncoming uniform flow, while in the second type of motion the number of relaxation zones will always be equal to the number of independent reactions. The restoration to equilibrium of a mixture which has been brought out of the equilibrium state by a forward shock requires the successive inclusion of all relaxation processes.

To illustrate the theory explained above, we performed calculations of the structure of the relaxation zones behind a shock wave in a mixture in which two reactions are taking place. In the calculations we set $\varepsilon_a^2/(\varepsilon m_\infty) = 1$, $\delta_a^2/\varepsilon_a^2 = 2$, $\lambda_1 = 100$, $e_1 = 10$, $\lambda_2 = 1$, $e_2 = 1$, $\lambda_{f\infty} = 1$. The results are shown in Figs. 1-4, where the solid curves correspond to the exact numerical solution of the problem, while the dashed curves relate to the data of the asymptotic analysis. In constructing Figs. 1-4, we used the variables $V = v_\infty/\gamma_{e\infty}$, $Q_1 = \lambda_1 q_1/(\gamma_{e\infty} e_1)$, $Q_2 = \lambda_2 q_2/(\gamma_{e\infty} e_2)$, $\Omega_1 = -\omega_1/(\gamma_{e\infty} e_1)$, $\Omega_2 = -\omega_2/(\gamma_{e\infty} e_2)$. The coordinate $x_2 = \lambda_2 x = x$ was used as the independent variable in Figs. 1, 3, and 4, and the coordinate $x_1 = \lambda_1 x$ in Fig. 2. The selected example is characterized by the fact that the velocity of the particles immediately behind the shock front is equal to the local singly frozen and singly equilibrium velocity of sound.

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